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Note

Some Results on "Product-Weighted Lead Codes"

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A number of product-weighting schemes for lead codes are derived by means of continued fraction expansions. Lead codes occur in connection with the linear Ising model.

1. INTRODUCTION

Consider a sequence of length $2n + 1$ with elements in $\{0, 1,..., n\}$, beginning and ending with 0, and such that adjacent elements differ by exactly one. Such a sequence is called a *lead code* for a two-candidate ballot ending in a tie after $2n$ votes. The enumeration of such sequences occurs, for example, in connection with the linear Ising model where the sequences are each weighted by the product of their elements. Such a weighting scheme is an example of *product weighting* and, of course, the initial and terminal zeroes are excluded from the product. Recently Rosen [4] has shown that the sum of the weights of sequences of this type is a tangent number.

In this paper we consider the enumeration of such sequences under a general weighting scheme, and we show that the generating function for the general problem may be developed as a continued fraction. Rosen's result is obtained as a special case, and a method is given for obtaining analogous results. The main theorem is given in Section 3, while Section 4 contains a numerical example which illustrates the problem and the results. Clearly, it is sufficient to consider sequences of length $2n - 1$, with elements in $\{1, 2,..., n\}$.

2. PRELIMINARIES AND NOTATION

Let $\mathscr{D}_{ij}^{(n)}$ be the set of sequences over $\{1, 2,..., n\}$, beginning with i and ending with a j, such that adjacent elements differ by exactly one. The *type*, $\tau(\sigma)$,

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of a sequence σ is $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ where i_j is the number of occurrences of the element j in σ; $\mathbf{x}^{\mathbf{i}}$ denotes

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \qquad \text{where} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n),$$

is a set of indeterminates; $[\mathbf{x}^{\mathbf{i}}]\, f(x)$ denotes the coefficient of $\mathbf{x}^{\mathbf{i}}$ in the formal power series $f(\mathbf{x})$. The enumeration problem defined in Section 1 is equivalent to determining $W_n^{(2n-1)}(x)$, where $W_n^{(p)}(\mathbf{x})$ is the sum of the weights of sequences of length p in $\mathscr{D}_{11}^{(n)}$ when the element i is given weight x_i, for $i = 1, 2, \ldots, n$.

3. The Main Theorem

The following theorem gives the generating function under the general weighting scheme as a continued fraction, and conversely gives a combinatorial interpretation of continued fractions of this type.

Theorem 1. *Let* $(0, 1, f_2, f_3, \ldots)$ *and* $(1, 1, g_2, g_3, \ldots)$ *be sequences generated by* $u_{m+1} = u_m - x^2 x_m x_{m+1} u_{m-1}$. *Then*

(i) $W_n^{(p)}(x) = [x^p]\, x x_1 \dfrac{f_n}{g_n}$;

(ii) $W_n^{(p)}(x) = [x^p] \left(\dfrac{x}{x_1^{-1}} - \dfrac{x^2}{x_2^{-1}} - \dfrac{x^2}{x_3^{-1}} - \cdots \right)$ *where* $p \leqslant 2n - 1$.

(using the continued fraction notation).

Proof. (i) Let $A^{(n)}$ be the adjacency matrix for the graph on the vertex set $\{v_1, v_2, \ldots, v_n\}$ such that v_i and v_j are adjacent iff $|i - j| = 1$. Let $X_n = x \operatorname{diag}(x_1, x_2, \ldots, x_n)$, I_n be the $n \times n$ identity matrix, and let $[A]_{ij}$ and $\operatorname{cof}_{ij}(A)$ denote the (i, j)-element and its cofactor, respectively, in the matrix A. Since the elements of $\mathscr{D}_{11}^{(n)}$ are in $[1{:}\,1]$ correspondence with paths in the graph with origin and terminus at v_1 then

$$\sum_{\sigma \in \mathscr{D}_{11}^{(n)}} \mathbf{x}^{\tau(\sigma)} = [(I_n - X_n A^{(n)})^{-1} X_n]_{11}\,.$$

But $\operatorname{cof}_{11}(I_n - X_n A^{(n)}) = 0$ if $n = 0$ and 1 if $n = 1$, and satisfies the above recurrence equation for u_m, so $f_n = \operatorname{cof}_{11}(I_n - X_n A^{(n)})$. Similarly $g_n = \det(I_n - X_n A^{(n)})$ and (i) follows immediately.

(ii) Let $u'_m = (x_1 x_2 \cdots x_m x^m)^{-1} u_m$. Then u'_m satisfies $u'_{m+1} = x^{-1} x_{m+1}^{-1} u'_m - u'_{m-1}$ so, from Perron [3, p. 16]

$$\frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i} = \frac{x_1 (x_2 x_3 \cdots x_i)^2\, x_{i+1}}{g_i\, g_{i+1}}\, x_{i+1} \qquad \text{if} \quad i \geqslant 1.$$

But g_i is a polynomial in x with a nonzero constant term so

$$[x^p]\,\frac{f_i}{g_i} = [x^p]\,\frac{f_\infty}{g_\infty} \qquad \text{if } \; p \leqslant 2i - 1.$$

Now

$$\frac{f_\infty}{g_\infty} = (xx_1)^{-1}\left\{\frac{x}{x_1^{-1}} - \frac{x^2}{x_2^{-1}} - \frac{x^2}{x_3^{-1}} - \cdots\right\},$$

from Perron [3, p. 4] and the theorem follows. ∎

Remark 1. The number of sequences of type i in $\mathscr{D}_{11}^{(n)}$ is accordingly

$$[\mathbf{x}^{\mathbf{i}}]\left\{\frac{1}{x_1^{-1}} - \frac{1}{x_2^{-1}} - \frac{1}{x_3^{-1}} - \cdots\right\}.$$

Put $x = 1$ in Theorem 1(ii). ∎

Remark 2. f_n and g_n in their determinant form are called *continuants* (Perron [3, p. 12] and Muir [2, p. 516]). ∎

The following lemma reduces continued fractions to the form of the one which appears in Theorem 1. The lemma will be used to derive particular weighting schemes.

LEMMA 1. *Let a_i, b_i, and α_i be nonzero for $i = 1, 2, \ldots$, and let $\alpha_0 = \alpha_{-1} = 1$. Then*

$$\frac{1}{x_1^{-1}} - \frac{1}{x_2^{-1}} - \frac{1}{x_3^{-1}} - \cdots = \frac{\alpha_0}{b_1} - \frac{\alpha_1}{b_2} - \frac{\alpha_2}{b_3} - \cdots,$$

where

$$x_{2j} = b_{2j}^{-1} \prod_{i=0}^{j-1} \frac{\alpha_{2i+1}}{\alpha_{2i}} \qquad \text{and} \qquad x_{2j+1} = b_{2j+1}^{-1} \prod_{i=0}^{j} \frac{\alpha_{2i}}{\alpha_{2i-1}}.$$

Proof. See Chrystal [1; Vol II; Chap. XXXIV; Sect. 10]. ∎

4. PARTICULAR WEIGHTING SCHEMES

A number of results for product weighted lead codes are now immediate. The following corollary is well known.

COROLLARY 1. *The number of sequences of length $2p - 1$ in $\mathscr{D}_{11}^{(n)}$ is the Catalan number $(1/p)\binom{2p-2}{p-1}$ if $p \leqslant n$.*

Proof. From Theorem 1(ii) the number of such sequences is

$$W_n^{2p-1}(1, 1, \dots, 1) = [x^{2p-1}]\, T, \quad \text{where } T = \frac{x}{1}\,\frac{}{-}\,\frac{x^2}{1}\,\frac{}{-}\,\frac{x^2}{1}\,\frac{}{-}\cdots = \frac{x}{1 - xT}.$$

The result follows from the Lagrange theorem for implicit functions. ▮

COROLLARY 2. *If the element i is given weight $(2i-1)^{-1}$, $i = 1, 2, \dots, n$ in a product weighting scheme then the sum of weights of sequences in $\mathscr{D}_{11}^{(n)}$ of length $2p-1$ is $T_{2p-1}/(2p-1)!$, for $p \leqslant n$, where T_j is a tangent number.*

Proof. From Theorem 1(ii), the number of such sequences is

$$W_n^{(2p-1)}\left(1, \frac{1}{3}, \frac{1}{5}, \dots\right) = [x^{2p-1}] \left\{ \frac{x}{1}\,\frac{}{-}\,\frac{x^2}{3}\,\frac{}{-}\,\frac{x^2}{5}\,\frac{}{-}\cdots \right\} = [x^{2p-1}] \tan x,$$

from Perron [3, p. 353], and the result follows. ▮

The following corollary is obtained by using the continued fraction representation of specific functions.

COROLLARY 3. *If* (i) 1 *has weight* 1,

(ii) $2i$ *has weight* $(1/(4i-1))\binom{2i-1}{i}^2(i^2/4^{2i-2})$, $i > 0$,

(iii) $2i+1$ *has weight* $(1/4i+1)\binom{2i-1}{i}^{-2}\, 4^{2i-1}$, $i \geqslant 0$,

in a product weighting scheme then the sum of the weights of sequences in $\mathscr{D}_{11}^{(n)}$ is $1/(2p-1)$ if $p \leqslant n$.

Proof. As a formal power series

$$\frac{1}{2}\log\frac{1+x}{1-x} = \frac{x}{1}\,\frac{}{-}\,\frac{1^2x^2}{3}\,\frac{}{-}\,\frac{2^2x^2}{5}\,\frac{}{-}\,\frac{3^2x^2}{7}\,\frac{}{-}\cdots, \quad \text{from Perron [3, p. 351].}$$

Thus the result follows by putting $\alpha_0 = 1$, $\alpha_i = i^2$ for $i \geqslant 1$ and $b_i = (2i-1)^{-1}$ in Lemma 1. The details are omitted since they are straightforward. ▮

Other results may be obtained in the same fashion by using other continued fractions. For example, from Perron [3, p. 348],

$$(1+x)^k = \frac{1}{1}\,\frac{}{-}\,\frac{kx}{1}\,\frac{}{+}\,\frac{1(1+k)\,x}{1\cdot 2}\,\frac{}{+}\,\frac{1(1-k)\,x}{2\cdot 3}$$
$$\frac{}{+}\,\frac{2(2+k)\,x}{3\cdot 4}\,\frac{}{+}\,\frac{2(2-k)}{4\cdot 5}\,\frac{}{+}\cdots$$

may be used in this way, and the case $k = \frac{1}{2}$ leads to a product weighting scheme in which the sum of the weights of sequences in $\mathscr{D}_{11}^{(n)}$ is a Catalan number. The details are omitted since they are straightforward. The following example demonstrates these results.

EXAMPLE 1. Consider the set $\mathscr{D}_{11}^{(4)}$ of sequences. From Theorem 1(ii) we have

$$W_4^{(7)}(x\mathbf{x}) = [x^7]\,\frac{x}{x_1^{-1}} - \frac{x^2}{x_2^{-1}} - \frac{x^2}{{}_3^{-1}} - \frac{x^2}{x_4^{-1}} = x_1{}^2x_2{}^2x_3{}^2x_4$$
$$+ 2x_1{}^3x_2{}^3x_3 + x_1{}^2x_2{}^3x_3{}^2 + x_1{}^4x_2{}^3.$$

The *types* of the sequences of length 7 in $\mathscr{D}^{(4)}$ are accordingly (2, 2, 2, 1), (3, 3, 1, 0), (2, 3, 2, 0), and (4, 3, 0, 0). The actual sequences, enumerated manually, are (1, 2, 3, 4, 3, 2, 1), (1, 2, 3, 2, 1, 2, 1), (1, 2, 1, 2, 3, 2, 1), (1, 2, 3, 2, 3, 2, 1), and (1, 2, 1, 2, 1, 2, 1). From Corollary 1, the number of such sequences is $\frac{1}{4}\binom{6}{3} = 5$. With product weighting scheme of Corollary 2 we have, by direct computation,

$$W_4^{(7)} = (3^2 \cdot 5^2 \cdot 7)^{-1} + 2(3^3 \cdot 5)^{-1} + (3^3 \cdot 5^2)^{-1} + (3^3)^{-1} = \frac{1}{7!}\,T_7$$
$$\text{where } T_7 = 272$$

is a tangent number. With the product weighting scheme of Corollary 3 we have

$$W_4^{(7)} = \left(\frac{1}{9}\cdot\frac{4^2}{5^2}\cdot\frac{9}{4\cdot 7}\right) + 2\left(\frac{1}{3^3}\cdot\frac{4}{5}\right) + \left(\frac{1}{3^3}\cdot\frac{4^2}{5^2}\right) + \frac{1}{3^3} = \frac{1}{7}. \quad \blacksquare$$

5. Rosen's Result

We now consider a final weighting scheme.

COROLLARY 4. *If the element i is given weight i, i* $= 1, 2,\ldots, n$ *in a product weighting scheme then the sum of the weights of sequences in* $\mathscr{D}_{11}^{(n)}$ *of length* $2p - 1$ *is* T_{2p-1}, *for* $p \leqslant n$, *where* T_j *is a tangent number.*

Proof. Let $F(t)$, $G(t)$, and $U(t)$ be the exponential generating functions for $\{f_i \mid i \geqslant 0\}$, $\{g_i \mid i \geqslant 0\}$, and $\{u_i \mid i \geqslant 0\}$ generated by $u_{m+1} = u_m - x_m x_{m+1} x^2 u_{m-1}$, where $x_m = m$ for this product weghting scheme. Let $F^{(n)}(t)$ and $G^{(n)}(t)$ denote the truncations of $F(t)$ and $(G(t)$ after the terms of degree n in t. Then

$$\frac{f_\infty}{g_\infty} = \lim_{n\to\infty}\lim_{t\to\infty}\frac{F^{(n)}(t)}{G^{(n)}(t)} = \lim_{t\to\infty}\frac{F(t)}{G(t)}.$$

Thus, from Theorem 1, we have $W_n^{(2p-1)} = [x^{2p-1}]\, x \lim_{t\to\infty} (F(t)/G(t))$ if $p \leqslant n$. Now $U(t)$ satisfies the differential equation

$$(1 + x^2t^2)\, U'(t) - (1 - 2tx^2)\, U(t) = -u_0 + u_1$$

whence

$$(1 + x^2t^2)\, U(t) \exp\{-x^{-1} \tan^{-1} xt\} = (u_1 - u_0) \int_0^t \exp\{-x^{-1} \tan^{-1} xs\}\, ds \text{ if}$$

$u_1 - u_0 \neq 0$ and $U(0)$ otherwise. But $F(t)$ and $G(t)$ satisfy the same differential equation as $U(t)$. Moreover, for $F(t)$ we have $f_1 - f_0 \neq 0$ while for $G(t)$ we have $g_1 - g_0 = 0$, and $G(0) = 1$. Thus

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = \int_0^\infty \exp\{-x^{-1} \tan^{-1} xt\}\, dt$$

Let $\tan xz = xt$, so

$$W_n^{(2p-1)} = [x^{2p-1}]\, x \int_0^\infty e^{-z} \sec^2 xz\, dz = [x^{2p-1}] \int_0^\infty e^{-z} \tan xz\, dz$$

and the result follows immediately. ∎

Remark 4. Rosen's result [4] is obtained with $p = n$. ∎

EXAMPLE 2. Consider the set $\mathscr{D}_{11}^{(4)}$ given in Example 1. With the product weighting scheme of Corollary 4, then

$$W_4^{(7)} = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4 + 2(1^3 \cdot 2^3 \cdot 3) + 1^2 \cdot 2^3 \cdot 3^2 + 1^4 \cdot 2^3 = T_7$$

where $T_7 = 272$

is a tangent number. The sequences of length 5 in $\mathscr{D}_{11}^{(4)}$ are (1, 2, 1, 2, 1) and (1, 2, 3, 2, 1) so, with the same product weighting scheme, then

$W_4^{(5)} = 1^3 \cdot 2^2 + 1^2 \cdot 2^2 \cdot 3 = T_5$ where $T_5 = 16$ is a tangent number. ∎

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REFERENCES

1. G. CHRYSTAL, "Textbook of Algebra," Vol. II, 7th ed. Chelsea, New York, 1964.
2. T. MUIR, "A Treatise on the Theory of Determinants" (revised by W. H. Metzler), Dover, New York, 1960.
3. O. PERRON, "Die Lehre von den Kettenbrüchen," Teubner, Leipzig/Berlin, 1929.
4. J. ROSEN, The number of product-weighted lead codes for ballots and its relation to the Ursell functions of the linear Ising model, *J. Combinatorial Theory Ser. A* **20** (1976), 377–384.